Mathematics: analysis and approaches

Standard Level

Paper 2

WORKED SOLUTIONS

1 hour 30 minutes

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[80 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

In an arithmetic sequence, $S_{30} = 1560$ and $u_{30} = 110$. Find the value of u_1 and the value of d.

Substituting into the formula $S_n = \frac{n}{2}(u_1 + u_n)$ gives:

$$1560 = \frac{30}{2}(u_1 + 110) \Rightarrow 104 = u_1 + 110 \Rightarrow u_1 = -6$$

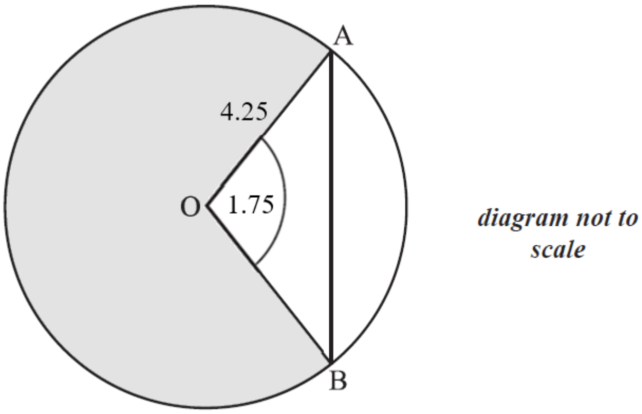
Substituting $u_1 = -6$ and $S_{30} = 1560$ into the formula $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ gives:

$$1560 = \frac{30}{2}(2\cdot(-6) + (30-1)d) \Rightarrow 104 = -12 + 29d$$

$$29d = 116 \Rightarrow d = 4$$

2. [Maximum mark: 7]

The circle shown below has center O and radius measuring 4.25 cm.



Points A and B lie on the circle and angle AOB measures 1.75 radians.

(a) Find AB. [3]

(b) Find the area of the shaded region. [4]

(a) using the cosine rule:

$$AB^2 = 4.25^2 + 4.25^2 - 2 \cdot 4.25 \cdot 4.25 \cdot \cos(1.75)$$

$$\approx 42.5641\ldots$$

$$AB = \sqrt{42.5641\ldots} \approx 6.5241\ldots$$

Thus, $AB \approx 6.52 \text{ cm}$

(b) angle corresponding to shaded sector $= 2\pi - 1.75$

area of shaded sector $= \frac{1}{2} r^2 \theta$

$$= \frac{1}{2} \cdot 4.25^2 \cdot (2\pi - 1.75)$$

$$\approx 40.9403\ldots$$

Thus, area of shaded sector $\approx 40.9 \text{ cm}^2$

3. [Maximum mark: 6]

A multiple-choice test consists of 12 questions. Each question has four answers from which to choose. Only one of the answers is correct. For each question, Boris randomly chooses one of the four answers.

(a) Write down the expected number of questions Boris answers correctly. [1]

(b) Find the probability that Boris answers exactly three questions correctly. [2]

(c) Find the probability that Boris answers more than three questions correctly. [3]

(a) Using formula for the mean of a binomial distribution:

$\mathrm{E}(X) = np = 12 \cdot 0.25 = 3$

(b) Let X be the random variable representing the number of questions Boris answers correctly

$X \sim \mathrm{B}(12, 0.25)$

Find $\mathrm{P}(X = 3)$ using GDC with $n = 12,\ p = 0.25$, lower bound $= 3$ and upper bound $= 3$

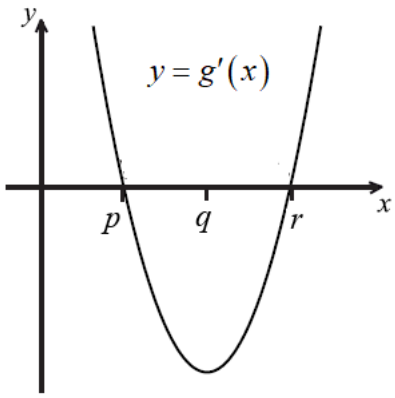
Thus, $\mathrm{P}(X = 3) \approx 0.258$

(c) Find $\mathrm{P}(X > 3)$ using GDC with $n = 12,\ p = 0.25$, lower bound $= 4$ and upper bound $= 12$

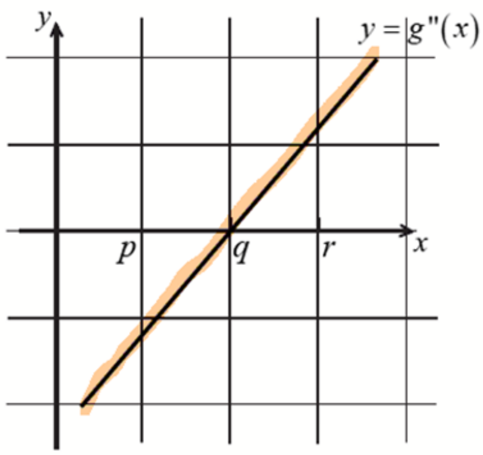
Thus, $\mathrm{P}(X > 3) \approx 0.351$

4. [Maximum mark: 6]

The diagram below shows part of the graph of the **gradient** function, $y = g'(x)$.



(a) On the grid below, sketch a graph of $y = g''(x)$, clearly indicating the x-intercept. [2]



The graph of $y = g'(x)$ is a parabola, so $g'(x) = ax^2 + bx + c,\ a, b, c \in \mathbb{R}$. Therefore, $g''(x) = 2ax + b$. Hence, the graph of $y = g''(x)$ will be a straight line.

The graph of $y = g''(x)$ corresponds to the gradient of the graph of $y = g'(x)$, which is negative on the interval $-\infty < x < q$, zero at $x = q$, and positive on the interval $q < x < \infty$.

(b) Complete the table below, for the graph of $y = g(x)$. [2]

	x-coordinate
(i) maximum point on g	$x = p$
(ii) minimum point on g	$x = r$

(c) Justify your answer to part (b) (ii). [2]

Minimum point on the graph of $y = g(x)$ exists where $g'(x) = 0$ and $g''(x) > 0$.

These conditions are satisfied at $x = r$.

5. [Maximum mark: 7]

Given that events A and B are independent, $\mathrm{P}(B) = 2\mathrm{P}(A)$, and $\mathrm{P}(A \cup B) = 0.72$, find $\mathrm{P}(B)$.

For independent events, $\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$

Therefore, the formula $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ becomes

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A) \cdot \mathrm{P}(B)$$

Substitute $\mathrm{P}(A \cup B) = 0.72$ and $\mathrm{P}(A) = \frac{1}{2}\mathrm{P}(B)$ into the equation above

$$0.72 = \frac{1}{2}\mathrm{P}(B) + \mathrm{P}(B) - \frac{1}{2}\mathrm{P}(B)\mathrm{P}(B)$$

$$\frac{1}{2}\left[\mathrm{P}(B)\right]^2 - \frac{3}{2}\mathrm{P}(B) + 0.72 = 0$$

Let $\mathrm{P}(B) = x$:

$$\frac{1}{2}x^2 - \frac{3}{2}x + 0.72 = 0$$

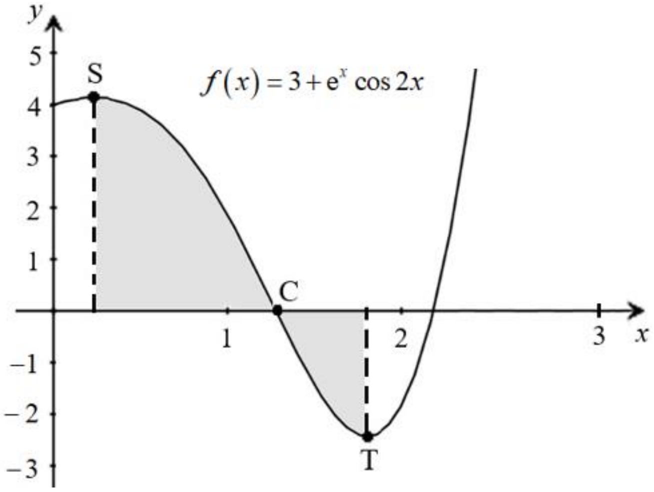
solve for x on GDC

$x = 0.6$; also $x = 2.4$ but $0 \le \mathrm{P}(B) \le 1$, hence $x \ne 2.4$

Thus, $\mathrm{P}(B) = 0.6$

6. [Maximum mark: 6]

Let $f(x) = 3 + e^{x}\cos 2x$, for $0 \le x \le 3$. A portion of the graph of f is shown below.



There is an x-intercept at the point C, a local maximum point at S where $x = s$, and a minimum point at T where $x = t$.

(a) Write down the following:

(i) the x-coordinate of C;

(ii) the value of s;

(iii) the value of t. [3]

(b) (i) Let $\int_s^t f(x)\,dx = k$. Calculate the value of k.

(ii) Explain why k is **not** the area of the shaded region. [3]

(a) (i) x-coordinate of C exists where $f(x) = 0$

$3 + e^{x}\cos 2x = 0$; solve for x on GDC

$x \approx 1.2792\ldots$

Thus, the x-coordinate of C is $x \approx 1.28$

(ii) Find value of s by analysing maximum points on the graph of $f(x)$ using GDC

$s \approx 0.232$

(iii) Find value of t by analysing minimum points on the graph of $f(x)$ using GDC

$t \approx 1.80$

Question 6 continues on the next page

Question 6 continued

(b) (i) Write down integral:

$$\int_{0.2318...}^{1.802...} \left(3+e^{x}\cos(2x)\right)dx = k$$

Evaluate the integral on GDC

$$\int_{0.2318...}^{1.802...} \left(3+e^{x}\cos(2x)\right)dx \approx 2.0912...$$

Thus, $k \approx 2.09$

(ii) The region between f and the x-axis from $x = s$ to $x = c$ is **above** the x-axis and the region between f and the x-axis from $x = c$ to $x = t$ is **below** the x-axis. Hence, the definite integral from $x = s$ to $x = c$ is a **positive** number and the definite integral from $x = c$ to $x = t$ is a **negative** number. Therefore, since k is the sum of the two definite integrals then the value of k is less than the total area of the shaded region.

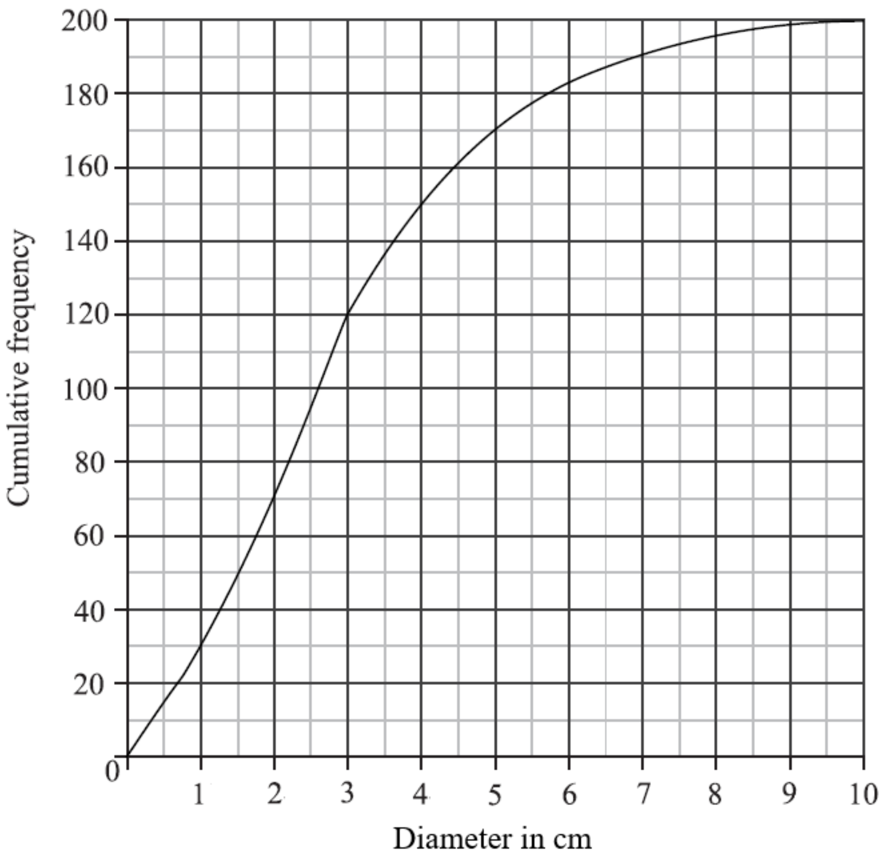
Section B

7. [Maximum mark: 15]
A farmer has an operation growing button mushrooms indoors that are sold at a local market. On a particular day, the farmer harvests 200 button mushrooms and measures the diameter (d) of each mushroom in centimeters. The results are shown in the frequency table below.

diameter, d cm	$0 < d \le 1$	$1 < d \le 2$	$2 < d \le 3$	$3 < d \le 4$	$4 < d \le 6$	$6 < d \le 7.5$	$7.5 < d \le 10$
frequency	30	40	50	30	33	11	6

(a) Calculate an estimate for the mean of the diameters of the mushrooms. [3]

(b) A cumulative frequency graph is given below for the diameters of the mushrooms.



Use the graph to answer the following.

(i) Estimate the interquartile range.

(ii) Given that 20% of the mushrooms have a diameter more than k cm, find the value of k. [6]

Question 7 continues on the next page

Question 7 continued

In preparation for selling the mushrooms, the farmer classifies each of them as *small*, *medium* or *large* using the following criteria.

Small: diameter is less than 2 cm
Medium: diameter is greater than or equal to 2 cm but less than 6 cm
Large: diameter is greater than or equal to 6 cm

(c) Write down the probability that a mushroom randomly selected from the day's harvest is *Small*. [2]

The cost of a *Small* mushroom is \$0.10, a *Medium* mushroom is \$0.15 and a *Large* mushroom is \$0.25.

(d) Copy and complete the table below which is the probability distribution for the cost \$$X$. [2]

Cost \$$X$	0.10	0.15	0.25
$\mathrm{P}(X = x)$		0.565	

(e) Find $\mathrm{E}(X)$. [2]

Worked Solution:

(a) mean $= \dfrac{\sum_{i=1}^{7}(\bar{d}_i \cdot f_i)}{n}$, where $\bar{d}_i$ is the midpoint of the ith diameter interval, f_i is the interval's corresponding frequency, and n is the total number of mushrooms

$$\text{mean} = \frac{0.5\cdot30+1.5\cdot40+2.5\cdot50+3.5\cdot30+5\cdot33+6.75\cdot11+8.75\cdot6}{200} \approx 2.9837\ldots \approx 2.98 \text{ cm}$$

(b) (i) $\mathrm{IQR} = \mathrm{Q3} - \mathrm{Q1}$

Q1 is diameter corresponding to a cumulative frequency of $\frac{200}{4} = 50$; from the graph, $\mathrm{Q1} = 1.5$

Q3 is diameter corresponding to a cumulative frequency of $3 \cdot \frac{200}{4} = 150$; from the graph, $\mathrm{Q3} = 4$

Therefore, $\mathrm{IQR} = 4 - 1.5 = 2.5$

(ii) 20% of 200 is 40; $200 - 40 = 160$; so looking for the diameter which 160 of the mushrooms have less than. The diameter corresponding to a cumulative frequency of 160 is $4.5\,\text{cm}$. Thus, $k = 4.5$

solution for question 7 continues on next page

7. solution continued

(c) $\mathrm{P}(\text{small}) = \mathrm{P}(\text{diameter} < 2)$; a diameter of $2\,\text{cm}$ corresponds to a cumulative frequency of 70

$\frac{70}{200} = 0.35$; Thus, $\mathrm{P}(\text{small}) = 0.35$

(d)

Cost $\$X$	0.10	0.15	0.25
$\mathrm{P}(X = x)$	0.35	0.565	0.085

$\mathrm{P}(X = 0.10) = \mathrm{P}(\text{small}) = 0.35$

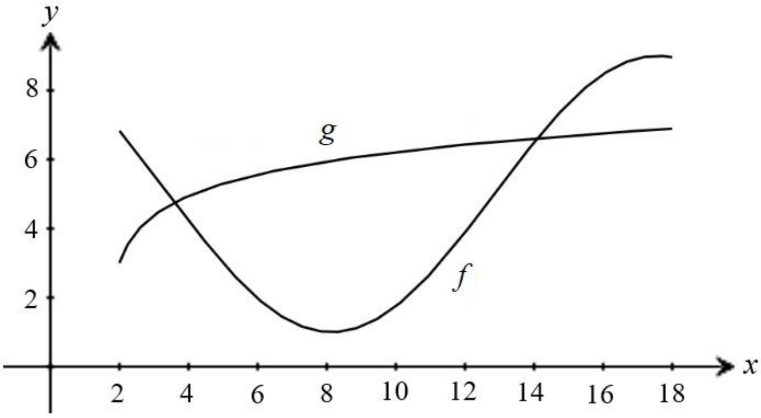
$\mathrm{P}(X = 0.25) = 1 - (\mathrm{P}(X = 0.10) + \mathrm{P}(X = 0.15)) = 1 - (0.35 + 0.565) = 0.085$

(e) $\mathrm{E}(X) = \sum x\mathrm{P}(X = x) = 0.10 \cdot 0.35 + 0.15 \cdot 0.565 + 0.25 \cdot 0.085$

Thus, $\mathrm{E}(X) \approx \$0.141$

8. [Maximum mark: 14]

Shown below are the graphs of $f(x)=4\sin\left(\frac{x}{3}+2\right)+5$ and $g(x)=\ln(3x-5)+3$, for $2\le x\le 18$.



(a) Calculate the area of the region enclosed by the graphs of f and g. [6]

(b) (i) Find $f'(x)$.

(ii) Find $g'(x)$. [4]

(c) There are two values of x, in the interval $2\le x\le 18$, for which the gradient of f is equal to the gradient of g. Find both these values of x. [4]

Worked Solution:

(a) Find point(s) of intersection:

$4\sin\left(\frac{x}{3}+2\right)+5=\ln(3x-5)+3$; solve for x on GDC

$x\approx 3.604\ldots$ and $x\approx 14.099\ldots$

Hence, $f(x)$ and $g(x)$ intersect at the points where $x\approx 3.604\ldots$ and $x\approx 14.099\ldots$

From the diagram, $g(x)\ge f(x)$ on the interval $3.604\ldots\le x\le 14.099\ldots$; thus,

$$\text{area} = \int_{3.604\ldots}^{14.099\ldots}\left(\ln(3x-5)+3-\left(4\sin\left(\frac{x}{3}+2\right)+5\right)\right)dx$$

$$= \int_{3.604\ldots}^{14.099\ldots}\left(\ln(3x-5)-4\sin\left(\frac{x}{3}+2\right)-2\right)dx$$

Evaluate integral on GDC:

$$\int_{3.604\ldots}^{14.099\ldots}\left(\ln(3x-5)-4\sin\left(\frac{x}{3}+2\right)-2\right)dx\approx 33.0498\ldots$$

Thus, area ≈ 33.0 units^2

solution for question 8 continues on next page

8. solution continued

(b) (i) $f(x)=4\sin\left(\frac{x}{3}+2\right)+5$ applying the chain rule

$$f'(x)=4\cos\left(\frac{x}{3}+2\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{3}+2\right)=4\cos\left(\frac{x}{3}+2\right)\cdot\frac{1}{3}$$

$$f'(x)=\frac{4}{3}\cos\left(\frac{x}{3}+2\right)$$

(ii) $g(x)=\ln(3x-5)+3$ applying the chain rule

$$g'(x)=\frac{1}{3x-5}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(3x-5)=\frac{1}{3x-5}\cdot 3$$

$$g'(x)=\frac{3}{3x-5}$$

(c) equating $f'(x)$ and $g'(x)$

$\frac{4}{3}\cos\left(\frac{x}{3}+2\right)=\frac{3}{3x-5}$; solve for x on GDC for the interval $2\le x\le 18$

$x\approx 8.4686\ldots$ and $x\approx 17.4191\ldots$

The two values of x in the interval $2\le x\le 18$ such that $f'(x)=g'(x)$ are $x\approx 8.47$ and $x\approx 17.4$

9. [Maximum mark: 13]

The heights of players in a basketball league are normally distributed with a mean of 188 cm (correct to three significant figures). It is known that 75% of the players have heights between 179 cm and 194 cm. The probability that a player is shorter than 179 cm is 0.05.

(a) Find the probability that a player is taller than 194 cm. [2]

(b) (i) Write down the standardized value, z, for 179 cm.

(ii) Hence, find the standard deviation of heights. [4]

To be invited to a special training camp, a player's height must be less than 1.5 standard deviations from the mean.

(c) A player is selected at random. Find the probability that the player is invited to the special training camp. [3]

30% of the players in the league are women. 75% of the women players are invited to the special training camp.

(d) Given that a player selected at random is invited to the special training camp, find the probability that the selected player is a woman. [4]

Worked Solution:

(a) Let X be the random variable representing the height of a basketball player

$$\mathrm{P}(X > 194) = 1 - (0.75 + 0.05) = 1 - 0.8 = 0.2$$

(b) (i) $\mathrm{P}(X < 179) = 0.05$; calculate z score on GDC with area $= 0.05$, $\mu = 0$ and $\sigma = 1$

$z = -1.6448\ldots$

Thus, the z score for 179 cm is $z \approx -1.64$

(ii) Using the formula for the standardized normal variable $z = \dfrac{x-\mu}{\sigma}$

substituting $x = 179\,\text{cm}$, $\mu = 188\,\text{cm}$ and $z = -1.644\ldots$ gives

$$-1.644\ldots = \frac{179-188}{\sigma} \Rightarrow \sigma = \frac{179-188}{-1.644\ldots} \approx 5.4716\ldots$$

Thus, the standard deviation of heights is $\sigma \approx 5.47\,\text{cm}$

solution for question 9 continues on next page

9. solution continued

(c) $1.5\sigma = 1.5 \cdot (5.4716\ldots) = 8.2074\ldots$

$188 + 8.2074\ldots = 196.207\ldots$

$188 - 8.2074\ldots = 179.793\ldots$

Find $\mathrm{P}(179.793\ldots < X < 196.207\ldots)$, using GDC with lower bound $= 179.207\ldots$, upper bound $= 196.207\ldots$, $\mu = 188$ and $\sigma = 5.4716\ldots$

$\mathrm{P}(179.793\ldots < X < 196.207\ldots) \approx 0.8663\ldots$

Thus, the probability of the player being invited to the special training camp is approximately 0.866

(d) Let T denote the event that a player is invited to training camp, and let W denote the event that a player is a woman

30% of players are women $\Rightarrow$ $\mathrm{P}(W) = 0.3$

75% of women players are invited to training camp $\Rightarrow$ $\mathrm{P}(T \mid W) = 0.75$

from part (c), $\mathrm{P}(T) \approx 0.866\ldots$

we are told to find $\mathrm{P}(W \mid T)$

Using conditional probability formula:

$\mathrm{P}(T \mid W) = \dfrac{\mathrm{P}(T \cap W)}{\mathrm{P}(W)}$ $\Rightarrow$ $\mathrm{P}(T \cap W) = \mathrm{P}(W)\mathrm{P}(T \mid W)$; substitute $\mathrm{P}(W) = 0.3$ and $\mathrm{P}(T \mid W) = 0.75$

$\mathrm{P}(T \cap W) = 0.3 \cdot 0.75 = 0.225$

Using conditional probability formula again:

$\mathrm{P}(W \mid T) = \dfrac{\mathrm{P}(T \cap W)}{\mathrm{P}(T)}$; substitute $\mathrm{P}(T \cap W) = 0.225$ and $\mathrm{P}(T) \approx 0.866\ldots$

$\mathrm{P}(W \mid T) = \dfrac{0.225}{0.866\ldots} \approx 0.2597\ldots$

Thus, $\mathrm{P}(W \mid T) \approx 0.260$